

March 2014

# Determination of Gravitational Counterterms Near Four Dimensions from RG Equations

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## Abstract

The finiteness condition of renormalization gives a restriction on the form of the gravitational action. By reconsidering the Hathrell's RG equations for massless QED in curved space, we determine the gravitational counterterms and the conformal anomalies as well near four dimensions. As conjectured for conformal couplings in 1970s, we show that at all orders of the perturbation they can be combined into two forms only: the square of the Weyl tensor in  $D$  dimensions and

$$E_D = G_4 + (D - 4)\chi(D)H^2 - 4\chi(D)\nabla^2 H,$$

where  $G_4$  is the usual Euler density,  $H = R/(D - 1)$  is the rescaled scalar curvature and  $\chi(D)$  is a finite function of  $D$  only. The number of the dimensionless gravitational couplings is also reduced to two.  $\chi(D)$  can be determined order by order in series of  $D - 4$ , whose first several coefficients are calculated. It has a universal value of  $1/2$  at  $D = 4$ . The familiar ambiguous  $\nabla^2 R$  term is fixed. At the  $D \rightarrow 4$  limit, the conformal anomaly  $E_D$  just yields the combination  $E_4 = G_4 - 2\nabla^2 R/3$ , which induces Riegert's effective action.

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# 1 Introduction

Recently, it has become increasingly important to understand how to include gravity within the framework of quantum field theory, especially when we consider models of the early universe such as inflation and quantum gravity. How to handle gravitational divergences is one of the most significant problems in this area.

We here consider gravitational counterterms for a four-dimensional quantum field theory in curved spacetime. Usually, we consider three independent gravitational counterterms and associated three dimensionless coupling constants. For a conformally coupled theory, however, there is an old conjecture in 1970s that gravitational divergences are simply renormalized by using conformally invariant counterterms: the square of the Weyl tensor and the Euler density [1, 2, 3, 4].

At a later time, however, the seemingly negative result that the  $R^2$ -divergence appears in calculations of 3-loop or more using dimensional regularization was reported by Brown and Collins [5] and then by Hathrell [6, 7] and Freeman [8]. On the other hand, Hathrell also showed in his paper that two counterterms of the Euler density and  $R^2$  are related to each other through renormalization group (RG) equations.

In this paper, we revive these old works and reconsider the meaning of their RG equations. It is then revealed that the appearance of the  $R^2$  divergence is simply a dimensional artifact coming from the fact that there is an indefiniteness in  $D$ -dimensional gravitational counterterms that reduce to conformally invariant ones at four dimensions. We see that the Euler density and  $R^2$  counterterms can be unified and the number of gravitational couplings can be reduced for conformally coupled theories. At the same time, the ambiguous term  $\nabla^2 R$  in conformal anomaly can be fixed completely.

In four dimensions, the conformal anomaly is obtained by regularizing the divergent quantity  $\delta^4(0) = \langle x|x' \rangle|_{x' \rightarrow x}$  coming from the path integral measure. On the other hand, if we use dimensional regularization, the result is independent of how to choose the measure because of  $\delta^D(0) = \int d^D k = 0$ .

This fact suggests that in dimensional regularization the information of the measure is contained between  $D$  and 4 dimensions. Thus, it is significant to determine the  $D$ -dependence of the counterterms. It is one of the aims of this study as well.

## 2 QED in Curved Space

As a prototype of conformally coupled quantum field theory, we here consider massless QED in curved space, because it is the simplest theory that the coupling between quantum fields and gravity is fixed unambiguously.

To begin with, we define the theory using dimensional regularization and summarize the notation and conventions. The action of QED in curved space is defined by

$$S = \int d^D x \sqrt{g} \left\{ \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + i \bar{\psi}_0 \not{D} \psi_0 + \frac{1}{2\xi_0} (\nabla^\mu A_{0\mu})^2 + a_0 F_D + b_0 G_4 + c_0 H^2 \right\},$$

where we consider the Wick-rotated Euclidean space. The quantity with the subscript 0 denotes the bare quantity before renormalization. The Dirac operator is defined by  $\not{D} = e^{\mu\alpha} \gamma_\alpha D_\mu$ , where  $e_\mu^a$  are vierbein fields in  $D$  dimensions satisfying  $e_\mu^a e_{\nu a} = g_{\mu\nu}$  and  $e_{\mu a} e^\mu_b = \delta_{ab}$ . The Dirac matrices are normalized as  $\{\gamma_a, \gamma_b\} = -2\delta_{ab}$ . The covariant derivative acting fermions is defined by  $D_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \Sigma^{ab} + ie_0 A_{0\mu}$ , where the connection 1-form and Lorentz generators are given by  $\omega_{\mu ab} = e^\nu_a (\partial_\mu e_{\nu b} - \Gamma^\lambda_{\mu\nu} e_{\lambda b})$  and  $\Sigma^{ab} = -\frac{1}{4} [\gamma^a, \gamma^b]$ , respectively.

For the moment, we consider the three types of gravitational counterterms adopted by Hathrell in his original paper [7]. The terms  $F_D$  is the square of the Weyl tensor in  $D$  dimensions defined by

$$F_D = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - \frac{4}{D-2} R_{\mu\nu} R^{\mu\nu} + \frac{2}{(D-1)(D-2)} R^2. \quad (2.1)$$

The term  $G_4$  is the Euler density and  $H$  is the scalar curvature scaled by a  $D$ -dependent factor, respectively, as

$$G_4 = R_{\mu\nu\lambda\sigma}^2 - 4R_{\mu\nu}^2 + R^2, \quad H = \frac{R}{D-1}.$$

Our sign convention of  $a_0$ ,  $b_0$  and  $c_0$  is different from [7]. In the later sections, we will show that the last two counterterms can be combined into one at last by relating  $b_0$  and  $c_0$ . On the other hand,  $a_0$  does not mix with the others.

The renormalization factors for quantum fields are defined by

$$A_{0\mu} = Z_3^{1/2} A_\mu, \quad \psi_0 = Z_2^{1/2} \psi$$

and the renormalizations of the coupling constant and the gauge-fixing parameter are defined by

$$e_0 = \mu^{2-D/2} Z_3^{-1/2} e, \quad \xi_0 = Z_3 \xi.$$

Here, the Ward-Takahashi identity  $Z_1 = Z_2$  is used.  $\mu$  is an arbitrary mass scale to make up the loss of mass dimensions and thus the renormalized coupling  $e$  is dimensionless. In the following, we mainly use the fine structure constant defined by  $\alpha = e^2/4\pi$ .

The RG equations are derived from the fact that bare quantities are independent of the arbitrary mass scale  $\mu$  such as

$$\mu \frac{d}{d\mu} (\text{bare}) = 0, \quad \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \mu \frac{d\alpha}{d\mu} \frac{\partial}{\partial \alpha} + \mu \frac{d\xi}{d\mu} \frac{\partial}{\partial \xi} + \cdots.$$

First, we consider the following equation:

$$\mu \frac{d}{d\mu} \left( \frac{e_0^2}{4\pi} \right) = 0 = \frac{\mu^{4-D}}{Z_3} \alpha \left( 4 - D - \mu \frac{d}{d\mu} \log Z_3 + \frac{\mu}{\alpha} \frac{d\alpha}{d\mu} \right).$$

From this, the beta function for  $\alpha$  is defined as

$$\beta(\alpha, D) \equiv \frac{1}{\alpha} \mu \frac{d\alpha}{d\mu} = (D - 4) + \bar{\beta}(\alpha),$$

where  $\bar{\beta} = d(\log Z_3)/d\mu$ . If we expand the renormalization factor as  $\log Z_3 = \sum_{n=1}^{\infty} f_n(\alpha)/(D - 4)^n$ ,  $\bar{\beta}$  is determined to be  $\alpha \partial f_1 / \partial \alpha$  and the equation  $\partial f_{n+1} / \partial \alpha + \bar{\beta} \partial f_n / \partial \alpha = 0$  must be satisfied in order that the beta function is finite when the  $D \rightarrow 4$  limit is taken.

In the following, we must be aware of the difference between  $\beta$  and  $\bar{\beta}$ , because  $1/\bar{\beta}$  is finite, while

$$\frac{1}{\beta} = \frac{1}{D-4} \left( 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\beta})^n}{(D-4)^n} \right) \quad (2.2)$$

has poles in the expansion for  $\alpha$ .

The gravitational counterterms are defined by

$$\begin{aligned} a_0 &= \mu^{D-4} (a + L_a), & L_a &= \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{(D-4)^n}, \\ b_0 &= \mu^{D-4} (b + L_b), & L_b &= \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{(D-4)^n}, \\ c_0 &= \mu^{D-4} (c + L_c), & L_c &= \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{(D-4)^n}, \end{aligned}$$

where  $L_{a,b,c}$  are the pure-pole terms whose residues are the functions of  $\alpha$  only and  $a, b, c$  are the gravitational coupling constants. The beta functions for them are defined by

$$\beta_a(\alpha, D) \equiv \mu \frac{da}{d\mu} = -(D-4)a + \bar{\beta}_a(\alpha)$$

and similar expressions for  $b$  and  $c$ .

As in the case of  $\beta$ , from the conditions that the bare coupling  $a_0$  is independent of  $\mu$  and  $\bar{\beta}_a$  is finite, we obtain the expression  $\bar{\beta}_a(\alpha) = -\partial(\alpha a_1)/\partial\alpha$  and the equation

$$\frac{\partial}{\partial\alpha} (\alpha a_{n+1}) + \bar{\beta}_a \frac{\partial a_n}{\partial\alpha} = 0 \quad (2.3)$$

for  $n \geq 1$ . The similar equations also satisfy for  $b_n$  and  $c_n$ .

When we discuss the finiteness of the theory, various normal products, namely finite composite operators, are significantly used. The normal product of dimension 4 is constructed as a linear combination of all available composite operators of dimension less than or equal 4 with appropriate symmetry and have to reduce to the bare field in the vanishing coupling limit.

For example,  $[F_{\mu\nu}F^{\mu\nu}] = (1 + \sum \text{poles})F_{0\mu\nu}F_0^{\mu\nu} + (\sum \text{poles})(\text{other operators})$ , where the notation  $[ ]$  denotes the normal product. The derivation of this normal product is briefly summarized in Appendix A.

The trace of the energy-momentum tensor denoted by  $\theta$ , which is intrinsically in a bare quantity obtained by applying  $\delta/\delta\Omega = (2/\sqrt{g})g_{\mu\nu}\delta/\delta g_{\mu\nu}$  to the action, can be written in a finite expression using the normal products as

$$\begin{aligned}\theta &= \frac{D-4}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{D-1}{2}E_{0\psi} + (D-4)(a_0F_D + b_0G_4 + c_0H^2) - 4c_0\nabla^2H \\ &= \frac{\beta}{4}[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2}(D-1+\bar{\gamma}_2)[E_\psi] \\ &\quad - \mu^{D-4}(\beta_aF_D + \beta_bG_4 + \beta_cH^2) - 4\mu^{D-4}(c-\sigma)\nabla^2H,\end{aligned}\tag{2.4}$$

where  $\bar{\gamma}_2 = \gamma_2 - (D-4)\xi\partial(\log Z_2)/\partial\xi$  is a combination that becomes independent of  $\xi$  and  $\gamma_2 = \mu d(\log Z_2)/d\mu$  is the usual anomalous dimension. The normal product  $[E_\psi]$  is the equation-of-motion operator for fermions defined in Appendix B. From the finiteness of the energy-momentum tensor, the  $\alpha$ -dependent function  $\sigma$  in the last term is determined to be  $\bar{\beta}_c + \sigma\bar{\beta} = 0$  and  $L_\sigma$  in (A.2) becomes equal to  $L_c$ . This is the expression of the conformal anomaly derived by Hathrell.

This expression, however, has the following undesirable structure. Taking the  $D \rightarrow 4$  limit, we can see that the dependence on the unspecified parameters  $\mu$ ,  $a$  and  $b$  in (2.4) disappears, but  $c$  in the last term remains with a finite effect, which is known as the ambiguous  $\nabla^2 R$  term in the conformal anomaly. One of the aims of this paper is to remove such an ambiguity and express the conformal anomaly in a simpler form.

### 3 Hathrell's RG Equations

In this section, we briefly review the Hathrell's RG equations [7], which are derived on the basis of the RG analysis by Brown and Collins [5] combined with a study of renormalized composite operators to deduce relationship between various quantities in the theory.

### 3.1 Two-point functions

We first consider the two-point function of the energy-momentum tensor modified as

$$\bar{\theta} = \theta - \frac{1}{2}(D-1)[E_\psi].$$

Since one-point functions are dimensionally regularized to zero for a massless theory in flat space,  $\langle [E_\psi(x)]P(y) \rangle_{\text{flat}} = \langle \delta P(y)/\delta\chi(x) \rangle_{\text{flat}} = 0$  is satisfied for a polynomial composite  $P(y)$  in the fields  $\bar{\psi}(y)$  and  $\psi(y)$ , where the functional derivative  $\delta/\delta\chi$  is defined in Appendix B. Using this fact and the condition of the two-point function of  $\theta$  (C.1) given in Appendix C, we obtain the following condition:

$$\langle \bar{\theta}(p)\bar{\theta}(-p) \rangle_{\text{flat}} - 8p^4\mu^{D-4}L_c = \text{finite} \quad (3.1)$$

in momentum space.

Next, we consider the following composite operator in flat space:

$$\begin{aligned} \{A^2\} &= \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} \\ &= \frac{1}{4} [F_{\mu\nu} F^{\mu\nu}] + \frac{\bar{\gamma}_2}{2\beta} [E_\psi]. \end{aligned}$$

This field is related to the trace of energy-momentum tensor as  $\bar{\theta}|_{\text{flat}} = \beta\{A^2\}$ , up to the term of gauge-fixing origin which is disregarded because it gives a vanishing contribution in physical correlation functions [8]. Note that  $\bar{\theta}$  is finite, while  $\{A^2\}$  is not so due to the presence of the last term with  $1/\beta$  in the second line.

The two-point function of  $\{A^2\}$  is denoted by  $\Gamma_{AA}(p^2) = \langle \{A^2(p)\} \{A^2(-p)\} \rangle_{\text{flat}}$  in momentum space. Here, although the composite operator  $\{A^2\}$  is not finite, the contribution from the term with  $1/\beta$  vanishes due to the property of  $[E_\psi]$ . Therefore,  $\Gamma_{AA}$  is given by the two-point function of the normal product  $[F_{\mu\nu} F^{\mu\nu}]$ . In such a correlation function, non-local divergences are canceled out and thus it can be written in the form

$$\Gamma_{AA}(p^2) - p^4\mu^{D-4} \left( \frac{D-4}{\beta} \right)^2 L_x = \text{finite}, \quad L_x = \sum_{n=1}^{\infty} \frac{x_n(\alpha)}{(D-4)^n}. \quad (3.2)$$

Here, the pure-pole term  $L_x$  is defined by this equation. The factor before  $L_x$  is introduced for the later convenience. The residue  $x_1$  will be directly calculated later.

Since  $\beta^2\Gamma_{AA} = \langle\bar{\theta}\bar{\theta}\rangle_{\text{flat}}$ , we can see that combining (3.1) and (3.2), the pure-pole terms satisfy the relation

$$(D-4)^2L_x - 8L_c = \text{finite}. \quad (3.3)$$

From this, we obtain the relationship between the residues,

$$c_n = \frac{1}{8}x_{n+2}. \quad (3.4)$$

This relation means that if the residue  $x_3$  is calculated, we can see the residue  $c_1$  and then obtain the general  $c_n$  from the RG equation (2.3).

So, we next derive the RG equation that relate  $x_3$  with  $x_1$ . Here, we use the fact that if  $F$  is a finite quantity,  $\beta^{-n}\mu d(\beta^n F)/d\mu$  is also finite in spite of the presence of the pole factor  $\beta^{-n}$  because of  $\beta^{-n}\mu d\beta^n/d\mu = n\alpha\partial\bar{\beta}/\partial\alpha$ . Applying this fact to the finite equation (3.2), we obtain

$$\frac{1}{\beta^2}\mu\frac{d}{d\mu}\left\{\beta^2\Gamma_{AA}(p^2) - p^4\mu^{D-4}(D-4)^2L_x\right\} = \text{finite}.$$

Since  $\beta\{A^2\}$  can be described in bare quantities, it satisfies  $\mu d(\beta\{A^2\})/d\mu = 0$  such that the first term vanishes. Thus, we obtain the RG equation

$$\frac{1}{\beta^2}\mu\frac{d}{d\mu}\left\{\mu^{D-4}(D-4)^2L_x\right\} = \text{finite}. \quad (3.5)$$

Expanding this equation and extracting the condition that poles cancel out, we obtain

$$\begin{aligned} \frac{\partial}{\partial\alpha}(\alpha x_2) - \frac{\bar{\beta}}{\alpha}\frac{\partial}{\partial\alpha}(\alpha^2 x_1) &= 0, \\ \frac{\partial}{\partial\alpha}(\alpha x_3) - \frac{\bar{\beta}}{\alpha}\frac{\partial}{\partial\alpha}(\alpha^2 x_2) + \frac{\bar{\beta}^2}{\alpha^2}\frac{\partial}{\partial\alpha}(\alpha^3 x_1) &= 0. \end{aligned} \quad (3.6)$$

Using these equations, we can derive the residues  $x_2$  and  $x_3$  from  $x_1$ . As is apparent from the relation (3.4), the equation of  $x_n$  for  $n \geq 3$  reduces to the same form as (2.3).



### 3.2 Three-point functions

Next, we consider the three-point function of the energy-momentum tensor. Here, we introduce new variable

$$\bar{\theta}_2(x, y) = \frac{\delta\bar{\theta}(x)}{\delta\Omega(y)} - \frac{1}{2}(D-1)\frac{\delta\bar{\theta}(x)}{\delta\chi(y)},$$

which satisfies the symmetric condition  $\bar{\theta}_2(x, y) = \bar{\theta}_2(y, x)$ . In terms of  $\bar{\theta}$  and  $\bar{\theta}_2$ , the condition of the three-point function of  $\theta$  (C.3) can be written in flat space as

$$\begin{aligned} & \langle \bar{\theta}(x)\bar{\theta}(y)\bar{\theta}(z) \rangle_{\text{flat}} - \langle \bar{\theta}(x)\bar{\theta}_2(y, z) \rangle_{\text{flat}} - \langle \bar{\theta}(y)\bar{\theta}_2(z, x) \rangle_{\text{flat}} - \langle \bar{\theta}(z)\bar{\theta}_2(x, y) \rangle_{\text{flat}} \\ & + \left\langle \frac{\delta^3 S}{\delta\Omega(x)\delta\Omega(y)\delta\Omega(z)} \right\rangle_{\text{flat}} = \text{finite}. \end{aligned}$$

The three-point function of  $\{A^2\}$  is denoted by  $\Gamma_{AAA}$ . Since  $\bar{\theta}|_{\text{flat}} = \beta\{A^2\}$  and  $\bar{\theta}_2(x, y)|_{\text{flat}} = -4\beta\{A^2\}\delta^D(x - y)$ , the condition above can be written in momentum space as

$$\begin{aligned} & \beta^3\Gamma_{AAA}(p_x^2, p_y^2, p_z^2) + 4\beta^2 \left\{ \Gamma_{AA}(p_x^2) + \Gamma_{AA}(p_y^2) + \Gamma_{AA}(p_z^2) \right\} \\ & + b_0 B(p_x, p_y, p_z) + c_0 C(p_x, p_y, p_z) = \text{finite}. \end{aligned}$$

The functions  $B$  and  $C$  are the contributions from the  $G_4$  and  $H^2$  terms in the action, respectively, which are defined by

$$\begin{aligned} B(p_x^2, p_y^2, p_z^2) &= -2(D-2)(D-3)(D-4) \\ &\quad \times \left[ p_x^4 + p_y^4 + p_z^4 - 2(p_x^2 p_y^2 + p_y^2 p_z^2 + p_z^2 p_x^2) \right], \\ C(p_x^2, p_y^2, p_z^2) &= -4 \left[ (D+2)(p_x^4 + p_y^4 + p_z^4) + 4(p_x^2 p_y^2 + p_y^2 p_z^2 + p_z^2 p_x^2) \right]. \end{aligned}$$

In the following, we consider the special cases that some momenta are taken to be on-shell. Combining (3.2) and (3.3), we obtain the equations,  $\beta^3\Gamma_{AAA}(p^2, p^2, 0) - 8(D-4)p^4\mu^{D-4}L_c = \text{finite}$ , and

$$\begin{aligned} & \beta^3\Gamma_{AAA}(p^2, 0, 0) - p^4\mu^{D-4} [2(D-2)(D-3)(D-4)L_b + 4(D-6)L_c] \\ & = \text{finite}. \end{aligned} \tag{3.7}$$

In general, removing the factor  $\beta^3$ ,  $\Gamma_{AAA}$  has the following form:

$$\begin{aligned} \Gamma_{AAA}(p_x^2, p_y^2, p_z^2) - \sum \text{poles} \times \{ \Gamma_{AA}(p_x^2) + \Gamma_{AA}(p_y^2) + \Gamma_{AA}(p_z^2) \} \\ - \mu^{D-4} \sum \text{poles} \times \{ \text{terms in } p_i^2 p_j^2 \} = \text{finite}. \end{aligned} \quad (3.8)$$

Since three-point functions with  $[E_\psi]$  do not vanish, the term  $[E_\psi]/\beta$  in  $\{A^2\}$  produces non-local poles because of the presence of  $1/\beta$ . Unlike  $\Gamma_{AA}$ ,  $\Gamma_{AAA}$  thus has non-local poles. The second term in (3.8) plays an important role to cancel out such non-local poles.

In order to determine the pure-pole factor in front of  $\Gamma_{AA}$  in (3.8), we consider the equation obtained by applying  $\alpha\partial/\partial\alpha$  to (3.2), which yields the equation for  $\Gamma_{AAA}(p^2, p^2, 0)$  because of  $\alpha\partial S/\partial\alpha|_{\text{flat}} = \int d^D x \{ -\{A^2\} + (D-4)[E_A]/2\beta - (\partial^\mu A_{0\mu})^2/2\xi_0 \}$  and  $\alpha\partial\{A^2\}/\partial\alpha = -(\alpha/\beta)(\partial\bar{\beta}/\partial\alpha)\{A^2\}$ . The pole factor can be extracted from this equation and fixed to be  $(\alpha^2/\beta)\partial(\bar{\beta}/\alpha)/\partial\alpha$ . Therefore,  $\Gamma_{AAA}(p^2, 0, 0)$  has the following form:

$$\Gamma_{AAA}(p^2, 0, 0) - \frac{\alpha^2}{\beta} \frac{\partial}{\partial\alpha} \left( \frac{\bar{\beta}}{\alpha} \right) \Gamma_{AA}(p^2) - p^4 \mu^{D-4} \left( \frac{D-4}{\beta} \right)^3 L_y = \text{finite}. \quad (3.9)$$

Here, the last pure-pole term  $L_y$  cannot be deduced from the equation for  $\Gamma_{AAA}(p^2, p^2, 0)$  mentioned above. This term is therefore defined through this equation, which is expanded as

$$L_y = \sum_{n=1}^{\infty} \frac{y_n(\alpha)}{(D-4)^n}.$$

Multiplying (3.9) by  $\beta^3$  and using (3.2) multiplied by  $\beta^2$  and (3.3), we obtain another equation including  $\beta^3\Gamma_{AAA}$  independent of (3.7). By eliminating  $\beta^3\Gamma_{AAA}$  from these equations, we obtain the following pole relation:<sup>2</sup>

$$\begin{aligned} 2(D-2)(D-3)(D-4)L_b + 4 \left[ D - 6 - 2\alpha^2 \frac{\partial}{\partial\alpha} \left( \frac{\bar{\beta}}{\alpha} \right) \right] L_c - (D-4)^3 L_y \\ = \text{finite}. \end{aligned} \quad (3.10)$$

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<sup>2</sup>We here correct the typo in [7] on the sign before  $2\alpha^2\partial(\bar{\beta}/\alpha)/\partial\alpha$  in (3.10) and the corresponding term in (3.9). It affects the calculations in Section 5.

Finally, we derive the RG equation for  $L_y$ . As similar to the derivation of (3.5), we consider the equation obtained by applying  $\beta^{-3}\mu d/d\mu$  to (3.9) multiplied by  $\beta^3$ . Noting that  $\mu d(\beta^3\Gamma_{AAA})/d\mu = \mu d(\beta^2\Gamma_{AA})/d\mu = 0$ , we obtain the following RG equation:

$$\left(\frac{D-4}{\beta}\right)^3 \left[ (D-4)L_y + \beta\alpha \frac{\partial}{\partial\alpha} L_y \right] + \alpha^2 \frac{\partial^2 \bar{\beta}}{\partial\alpha^2} \left(\frac{D-4}{\beta}\right)^2 L_x = \text{finite}, \quad (3.11)$$

where (3.2) is used.

## 4 Reconsiderations of Confomal Anomalies

Originally, Hathrell considered the three-type of gravitational couplings denoted by  $a$ ,  $b$  and  $c$ , as shown in the previous sections, and he concluded that the  $R^2$  divergence appears at  $o(\alpha^3)$  even for QED in curved space.

However, on the other hand, the derived equation (3.10) gives the relationship between the pure-pole terms  $L_b$  and  $L_c$  through (3.3) and (3.11). So, against his conclusion, his results rather indicate that the independent gravitational counterterms are only two. In this section, we reconsider his results in this context.

We here propose that the gravitational counterterms are given by the two terms as

$$S_g = \int d^D x \sqrt{g} \{a_0 F_D + b_0 G_D\}. \quad (4.1)$$

The novel term  $G_D$  is defined by

$$G_D = G_4 + (D-4)\chi(D)H^2, \quad (4.2)$$

where  $\chi(D)$  is a finite function of  $D$  only and thus this term reduces to the Euler density at  $D = 4$ .

By repeating the previous procedure using the counterterm (4.1) again, we can easily find that the finiteness conditions simply result in the Hathrell's RG equations, (3.3), (3.5), (3.10) and (3.11), under the relation

$$L_c - (D-4)\chi(D)L_b = \text{finite}. \quad (4.3)$$

The RG equations (2.3) for  $a_n$  and  $b_n$  are satisfied, and also for  $c_n$  through the relation (4.3), though there is no  $\beta_c$ . Thus, we can make the theory finite using two gravitational counterterms only. In the next section, we will show that the function  $\chi$  can be determined completely by solving the coupled RG equations order by order.

On the other hand, we have to pay more attention to the calculation of the finite quantities such as the expression of the conformal anomaly, because the counterterm (4.1) implies that the finite parameter  $c$  is eliminated, while extra finite terms are added.

According to the derivation briefly summarized in Appendix A, we find that the expression of the normal product  $[F_{\mu\nu}^2]$  in the case of (4.1) can be determined up to the total-divergence term as

$$\begin{aligned} \frac{1}{4}[F_{\mu\nu}F^{\mu\nu}] &= \frac{D-4}{4\beta}F_{0\mu\nu}F_0^{\mu\nu} - \frac{\bar{\gamma}_2}{2\beta}[E_\psi] + \frac{D-4}{\beta}\mu^{D-4}\left[\left(L_a + \frac{\bar{\beta}_a}{D-4}\right)F_D \right. \\ &\quad \left. + \left(L_b + \frac{\bar{\beta}_b}{D-4}\right)G_D - \frac{4\chi(D)(\sigma + L_\sigma)}{D-4}\nabla^2 H\right], \end{aligned} \quad (4.4)$$

where  $\sigma$  is a finite function of  $\alpha$  and  $L_\sigma = \sum_{n=1}^{\infty} \sigma_n/(D-4)^n$ , which will be determined below. The factor  $\chi$  in the last term is multiplied for convenience.

Using the expression of the normal product (4.4), the trace of the energy-momentum tensor can be expressed in a manifestly finite form as

$$\begin{aligned} \theta &= \frac{D-4}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{D-1}{2}E_{0\psi} + (D-4)\left[a_0F_D + b_0\left(G_D - 4\chi(D)\nabla^2 H\right)\right] \\ &= \frac{\beta}{4}[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2}(D-1+\bar{\gamma}_2)[E_\psi] - \mu^{D-4}(\beta_aF_D + \beta_bG_D) \\ &\quad - 4\mu^{D-4}\chi(D)[(D-4)b - \sigma + b_1]\nabla^2 H. \end{aligned} \quad (4.5)$$

Here, in the second equality, we use the consistency condition to make the last term finite such as

$$(D-4)(b + L_b) - (\sigma + L_\sigma) = \text{finite} = (D-4)b - \sigma + b_1.$$

The right-hand side just appears in the last term of the expression (4.5). The residue  $\sigma_n$  of  $L_\sigma$  is then determined using the residue of  $L_b$  as

$$\sigma_n = b_{n+1}$$

for  $n \geq 1$ .

Furthermore, in order to determine the finite value  $\sigma$ , we consider the finite quantity  $\beta^{-1}\mu d/d\mu(\beta[F_{\mu\nu}F^{\mu\nu}]/4)$ . Rewriting this expression using the fact that the energy-momentum tensor is independent of  $\mu$ , we obtain

$$\begin{aligned} \frac{1}{\beta}\mu\frac{d}{d\mu}\left(\frac{\beta}{4}[F_{\mu\nu}F^{\mu\nu}]\right) &= -\frac{1}{2}\alpha\frac{\partial\bar{\gamma}_2}{\partial\alpha}[E_\psi] + \mu^{D-4}\left(\alpha\frac{\partial\bar{\beta}_a}{\partial\alpha}F_D + \alpha\frac{\partial\bar{\beta}_b}{\partial\alpha}G_D\right) \\ &\quad + 4\mu^{D-4}\chi(D)\left\{\frac{D-4}{\beta}[(D-4)b - \sigma + b_1]\right. \\ &\quad \left.+ \frac{1}{\beta}\left[(D-4)\mu\frac{db}{d\mu} - \mu\frac{d\sigma}{d\mu} + \mu\frac{db_1}{d\mu}\right]\right\}\nabla^2 H \quad (4.6) \end{aligned}$$

From the condition that the last term is finite, we obtain

$$\sigma = \bar{\beta}_b + b_1$$

and then the inside of the bracket  $\{ \}$  reduces to the finite value  $-\alpha\partial\bar{\beta}_b/\partial\alpha$ .

Substituting this result into (4.5), we obtain the following simpler expression of conformal anomaly:

$$\theta = \frac{\beta}{4}[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2}(D-1+\bar{\gamma}_2)[E_\psi] - \mu^{D-4}(\beta_a F_D + \beta_b E_D), \quad (4.7)$$

where the quantity  $E_D$  is defined by

$$E_D = G_D - 4\chi(D)\nabla^2 H. \quad (4.8)$$

The  $G_D$  and  $\nabla^2 H$  terms in the right-hand side of the normal product (4.4) are also unified in this form.

The novel function  $E_D$  has a desirable property as the other conformal anomalies  $F_D$  and  $F_{\mu\nu}^2$  have, which is

$$\frac{\delta}{\delta\Omega}\int d^D x \sqrt{g} E_D = (D-4)E_D.$$

Here, the volume integral of  $E_D$  is nothing but the  $G_D$  counterterm.

## 5 Determination of Gravitational Counterterms

In this section, we explicitly solve the RG equations and determine the constant  $\chi$  order by order.

To determine the pole terms, we need the information of the QED beta function and the simple-pole residues of  $L_x$  and  $L_y$ . In this section, they are expanded as follows:

$$\begin{aligned}\bar{\beta} &= \beta_1\alpha + \beta_2\alpha^2 + \beta_3\alpha^3 + o(\alpha^4), \\ x_1 &= X_1 + X_2\alpha + X_3\alpha^2 + o(\alpha^3), \\ y_1 &= Y_1 + Y_2\alpha + Y_3\alpha^2 + o(\alpha^3).\end{aligned}$$

The specific values of these coefficients will be given in the next section.

We first calculate the residue  $x_n$ . Using the RG equations (3.6), we can derive  $x_2$  and  $x_3$  from  $x_1$ . Furthermore,  $x_n$  for  $n \geq 4$  can be derived from  $x_3$  using the fact that the RG equation of  $x_n$  reduces to the same form as (2.3) for  $n \geq 3$  as mentioned before. Using the expressions for  $\bar{\beta}$  and  $x_1$  above, we derive the expression of  $x_n$  to  $o(\alpha^{n+1})$  for each  $n$ . For the first several residues, we obtain

$$\begin{aligned}x_2 &= \beta_1 X_1 \alpha + \left(\frac{2}{3}\beta_2 X_1 + \beta_1 X_2\right) \alpha^2 + \left(\frac{1}{2}\beta_3 X_1 + \frac{3}{4}\beta_2 X_2 + \beta_1 X_3\right) \alpha^3 + o(\alpha^4), \\ x_3 &= -\frac{1}{12}\beta_1\beta_2 X_1 \alpha^3 + \left(-\frac{1}{15}\beta_2^2 X_1 - \frac{1}{10}\beta_1\beta_3 X_1 - \frac{1}{20}\beta_1\beta_2 X_2\right) \alpha^4 + o(\alpha^5), \\ x_4 &= \frac{1}{20}\beta_1^2\beta_2 X_1 \alpha^4 + \left(\frac{31}{360}\beta_1\beta_2^2 X_1 + \frac{1}{30}\beta_1^2\beta_2 X_2 + \frac{1}{15}\beta_1^2\beta_3 X_1\right) \alpha^5 + o(\alpha^6).\end{aligned}\tag{5.1}$$

Note that the lowest term of  $x_n$  is given by  $o(\alpha^{n-1})$  for  $n \leq 2$ , while for  $n \geq 3$  it is reduced to  $o(\alpha^n)$ . It is probably associated with the fact that the RG equation of  $x_n$  becomes simpler for  $n \geq 3$ . And also, the  $o(\alpha^3)$  term of  $x_2$  has the coefficient  $X_3$  of 3-loop origin, while the  $o(\alpha^{n+1})$  term of  $x_n$  for  $n \geq 3$  does not include this coefficient.

The residue  $c_n$  is also obtained through the relation  $c_n = x_{n+2}/8$  (3.4), and thus  $c_1$  starts from  $o(\alpha^3)$ .

Next, we calculate  $y_n$  to  $o(\alpha^{n+1})$  for each  $n$ . Expanding the RG equation (3.11) and evaluating the finiteness condition such that the  $n$ -th pole term cancels out, we can derive the following relationship between the residues:

$$\begin{aligned} & \frac{\partial}{\partial \alpha} (\alpha y_{n+1}) + \bar{\beta} \alpha \frac{\partial y_n}{\partial \alpha} \\ & + \sum_{m=1}^{n-1} (-1)^m \frac{(m+1)(m+2)}{2} \bar{\beta}^m \left[ \frac{\partial}{\partial \alpha} (\alpha y_{n-m+1}) + \bar{\beta} \alpha \frac{\partial y_{n-m}}{\partial \alpha} \right] \\ & + (-1)^n \frac{(n+1)(n+2)}{2} \bar{\beta}^n \frac{\partial}{\partial \alpha} (\alpha y_1) - \alpha^2 \frac{\partial^2 \bar{\beta}}{\partial \alpha^2} \sum_{m=1}^n (-1)^m m \bar{\beta}^{m-1} x_{n-m+1} = 0 \end{aligned} \quad (5.2)$$

for  $n \geq 1$ . Since we have already derived the residue  $x_n$  from  $x_1$ , we can derive the residue  $y_n$  from  $x_1$  and  $y_1$  using this equation. The first several residues are given by

$$\begin{aligned} y_2 &= \frac{3}{2} \beta_1 Y_1 \alpha + \left( -\frac{2}{3} \beta_2 X_1 + \beta_2 Y_1 + \frac{5}{3} \beta_1 Y_2 \right) \alpha^2 \\ &+ \left( -\frac{3}{2} \beta_3 X_1 - \frac{1}{2} \beta_2 X_2 + \frac{3}{4} \beta_3 Y_1 + \frac{5}{4} \beta_2 Y_2 + \frac{7}{4} \beta_1 Y_3 \right) \alpha^3 + o(\alpha^4), \\ y_3 &= \frac{1}{2} \beta_1^2 Y_1 \alpha^2 + \left( -\frac{2}{3} \beta_1 \beta_2 X_1 + \frac{5}{8} \beta_1 \beta_2 Y_1 + \frac{2}{3} \beta_1^2 Y_2 \right) \alpha^3 + \left( -\frac{3}{2} \beta_1 \beta_3 X_1 \right. \\ &\quad \left. - \frac{1}{2} \beta_1 \beta_2 X_2 - \frac{2}{5} \beta_2^2 X_1 + \frac{3}{4} \beta_1^2 Y_3 + \frac{59}{60} \beta_1 \beta_2 Y_2 + \frac{1}{5} \beta_2^2 Y_1 + \frac{9}{20} \beta_1 \beta_3 Y_1 \right) \alpha^4 \\ &\quad + o(\alpha^5), \\ y_4 &= \frac{1}{40} \beta_1^2 \beta_2 Y_1 \alpha^4 + \left( \frac{1}{30} \beta_1^2 \beta_3 Y_1 + \frac{13}{240} \beta_1 \beta_2^2 Y_1 + \frac{1}{90} \beta_1^2 \beta_2 Y_2 + \frac{13}{180} \beta_1 \beta_2^2 X_1 \right) \alpha^5 \\ &\quad + o(\alpha^6), \\ y_5 &= -\frac{1}{60} \beta_1^3 \beta_2 Y_1 \alpha^5 + \left( -\frac{53}{1260} \beta_1^2 \beta_2^2 X_1 - \frac{1}{42} \beta_1^3 \beta_3 Y_1 - \frac{89}{1680} \beta_1^2 \beta_2^2 Y_1 \right. \\ &\quad \left. - \frac{1}{126} \beta_1^3 \beta_2 Y_2 \right) \alpha^6 + o(\alpha^7). \end{aligned} \quad (5.3)$$

Note that the lowest term of  $y_n$  is given by  $o(\alpha^{n-1})$  for  $n \leq 3$ , while for  $n \geq 4$  it starts from  $o(\alpha^n)$ . And also, the  $o(\alpha^{n+1})$  term of  $y_n$  has the coefficient  $Y_3$  of 3-loop origin for  $n \leq 3$ , while for  $n \geq 4$  it does not appear. This result

seems to reflect the fact that for  $n = k + 3$  with  $k \geq 1$  the RG equation (5.2) reduces to the simpler form

$$\frac{\partial}{\partial \alpha} (\alpha y_{k+4}) + \bar{\beta} \alpha \frac{\partial y_{k+3}}{\partial \alpha} = -\alpha^2 \frac{\partial^2 \bar{\beta}}{\partial \alpha^2} (x_{k+3} + \bar{\beta} x_{k+2}),$$

as in the case of  $x_n$  for  $n \geq 3$ .

Now, we can solve the RG equation (3.10) under the relation (4.3). Expanding (3.10) and extracting the finiteness condition that poles cancel out, we obtain

$$4b_{n+1} + 6b_{n+2} + 2b_{n+3} - 8 \left[ 1 + \alpha^2 \frac{\partial}{\partial \alpha} \left( \frac{\bar{\beta}}{\alpha} \right) \right] c_n + 4c_{n+1} - y_{n+3} = 0 \quad (5.4)$$

for  $n \geq 1$ . Since  $y_n$  is related with  $c_n$  through (5.2) and (3.4), this equation connect  $b_n$  with  $c_n$ . Since the equation for  $n \geq 2$  can be derived from the  $n = 1$  equation using the other RG equations, we use the  $n = 1$  equation only below.

The  $D$ -dependent constant  $\chi$  is expanded as a power series in  $D - 4$  such as

$$\chi(D) = \sum_{n=1}^{\infty} \chi_n (D - 4)^{n-1} = \chi_1 + \chi_2 (D - 4) + \chi_3 (D - 4)^2 + \dots$$

The relation (4.3) is then expressed as

$$\begin{aligned} c_1 &= \chi_1 b_2 + \chi_2 b_3 + \chi_3 b_4 + \dots, \\ c_2 &= \chi_1 b_3 + \chi_2 b_4 + \chi_3 b_5 + \dots \end{aligned} \quad (5.5)$$

and so on. Since  $b_n$  ( $n \geq 3$ ) can be expressed by  $b_2$  using the RG equation (2.3) for  $b_n$ , this relation implies that  $c_n$  can be obtained from  $b_2$ .

Since  $c_1$  starts from  $o(\alpha^3)$ ,  $b_2$  also starts from  $o(\alpha^3)$ . For the moment,  $b_2$  is expanded as follows:

$$b_2 = B_1 \alpha^3 + B_2 \alpha^4 + B_3 \alpha^5 + o(\alpha^6).$$



From the RG equation (2.3) for  $b_n$ , we obtain the expressions

$$\begin{aligned} b_3 &= -\frac{3}{5}\beta_1 B_1 \alpha^4 - \left(\frac{1}{2}\beta_2 B_1 + \frac{2}{3}\beta_1 B_2\right) \alpha^5 + o(\alpha^6), \\ b_4 &= \frac{2}{5}\beta_1^2 B_1 \alpha^5 + o(\alpha^6), \\ b_5 &= o(\alpha^6) \end{aligned}$$

and so on.

Substituting these expressions into the RG equation (5.4) of  $n = 1$  and expanding up to  $\alpha^5$ , we obtain

$$\begin{aligned} &4(1 - 2\chi_1) B_1 \alpha^3 + \left\{4(1 - 2\chi_1) B_2 + \frac{6}{5}(-3 - 2\chi_1 + 4\chi_2) \beta_1 B_1\right\} \alpha^4 \\ &+ \left\{4(1 - 2\chi_1) B_3 + (-3 - 10\chi_1 + 4\chi_2) \beta_2 B_1 + \frac{4}{3}(-3 - 2\chi_1 + 4\chi_2) \beta_1 B_2\right. \\ &\quad \left. + \frac{4}{5}(1 + 2\chi_2 - 4\chi_3) \beta_1^2 B_1\right\} \alpha^5 - y_4(\alpha) = o(\alpha^6). \end{aligned} \quad (5.6)$$

Here, note that the residue  $y_4$  starts from  $o(\alpha^4)$ . Thus, from the vanishing condition at  $o(\alpha^3)$ , the coefficient  $\chi_1$  is determined to be

$$\chi_1 = \frac{1}{2}. \quad (5.7)$$

This is just the result found by Hathrell, which is expressed as  $b_2 = 2c_1 + o(\alpha^4)$  in his paper [7].

Since  $\chi_1 = 1/2$ , the  $B_2$ -dependence of  $o(\alpha^4)$  in (5.6) disappears. Therefore, substituting the explicit expression of  $y_4$  (5.3) into (5.6), we obtain the expression

$$\chi_2 = 1 + \frac{1}{192} \frac{\beta_1 \beta_2 Y_1}{B_1} \quad (5.8)$$

from the vanishing condition at  $o(\alpha^4)$ . Using the relation (5.5) and the result (5.7), we can derive

$$B_1 = -\frac{1}{48} \beta_1 \beta_2 X_1 \quad (5.9)$$

from the expression of  $c_1$  at  $o(\alpha^3)$  which can be read from (5.1) through the relation (3.4). Substituting this expression, we obtain

$$\chi_2 = 1 - \frac{Y_1}{4X_1}. \quad (5.10)$$

Using the result (5.10), the coefficient  $B_2$  can be calculated from the expression of  $c_1$  at  $o(\alpha^4)$  as

$$B_2 = -\frac{1}{160} \left( 4\beta_1^2\beta_2X_1 + \frac{8}{3}\beta_2^2X_1 + 4\beta_1\beta_3X_1 + 2\beta_1\beta_2X_2 - \beta_1^2\beta_2Y_1 \right). \quad (5.11)$$

Furthermore, since the  $B_3$ -dependence in (5.6) disappears due to (5.7), we can solve the condition (5.6) at  $o(\alpha^5)$  using the expressions of (5.9), (5.11) and (5.10). Thus, we obtain

$$\chi_3 = \frac{1}{8} \left( 2 - \frac{Y_1}{X_1} \right) \left( 3 - \frac{Y_1}{X_1} \right) - \frac{1}{6} \frac{\beta_2}{\beta_1^2} \left( 1 - \frac{Y_1}{X_1} \right) + \frac{1}{6} \frac{X_2}{\beta_1 X_1} \left( \frac{Y_2}{X_2} - \frac{3}{2} \frac{Y_1}{X_1} \right) \quad (5.12)$$

Here,  $\chi_3$  does not depend on  $\beta_3$ .

In this way, we can determine the coefficient  $\chi_n$  order by order.

## 6 Values of The Parameters

Let us determine the coefficients  $\chi_n$  and the residues of pole terms by substituting the concrete values. We here use the expression of the beta function up to 3-loop order computed as [9]

$$\beta_1 = \frac{8}{3} \frac{1}{4\pi}, \quad \beta_2 = 8 \frac{1}{(4\pi)^2}, \quad \beta_3 = -\frac{124}{9} \frac{1}{(4\pi)^3}. \quad (6.1)$$

The values of  $X_{1,2}$  and  $Y_{1,2}$  are obtained from the direct 2-loop computations of  $\Gamma_{AA}$  and  $\Gamma_{AAA}$ , respectively [7]. The function  $\Gamma_{AA}$  is calculated as

$$\begin{aligned} & \Gamma_{AA}(p^2)(2\pi)^D \delta^D(p+q) \\ &= \left( \frac{D-4}{\beta} Z_3 \right)^2 \frac{1}{4} \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} K^{\mu\nu}(k, k-p) K^{\lambda\sigma}(l, l-q) \\ & \quad \times \langle A_\mu(k) A_\nu(p-k) A_\lambda(l) A_\sigma(q-l) \rangle_{\text{flat}}, \end{aligned}$$

where

$$K^{\mu\nu}(k, k-p) = k \cdot (k-p) \delta^{\mu\nu} - (k-p)^\mu k^\nu.$$

The renormalization factor  $Z_3$  arises by replacing  $F_{0\mu\nu}$  in  $\{A^2\}$  with  $Z_3^{1/2} F_{\mu\nu}$ . The four-point function of  $A_\mu$  is evaluated up to  $o(\alpha)$  for the diagrams such that two composite operators are connected. Carrying out the momentum integrals, we obtain

$$\Gamma_{AA}(p^2) = \frac{p^4 \mu^{D-4}}{(4\pi)^2} \left\{ -\frac{1}{2} \frac{1}{D-4} + \frac{\alpha}{4\pi} \left( \frac{4}{3} \frac{1}{(D-4)^2} + \frac{5}{3} \frac{1}{D-4} \right) \right\} + \text{finite}.$$

From this expression, we obtain the coefficients of  $x_1$  as

$$X_1 = -\frac{1}{2} \frac{1}{(4\pi)^2}, \quad X_2 = \frac{5}{3} \frac{1}{(4\pi)^3}. \quad (6.2)$$

Taking account of the factor  $((D-4)/\beta)^2$  in (3.2) introduced for convenience, the lowest order term of  $x_2$  is also determined to be  $-4\alpha/3(4\pi)^3$ , which is consistent with the RG equation (3.6).

Similarly, the three-point function  $\Gamma_{AAA}$  with two on-shell momenta is calculated as

$$\Gamma_{AAA}(p^2, 0, 0) = \frac{p^4 \mu^{D-4}}{(4\pi)^2} \left\{ -\frac{1}{2} \frac{1}{D-4} + \frac{\alpha}{4\pi} \left( 2 \frac{1}{(D-4)^2} + \frac{11}{6} \frac{1}{D-4} \right) \right\} + \text{finite}.$$

From this expression, we obtain the coefficients of  $y_1$  as

$$Y_1 = -\frac{1}{2} \frac{1}{(4\pi)^2}, \quad Y_2 = \frac{11}{6} \frac{1}{(4\pi)^3}. \quad (6.3)$$

The lowest order term of  $y_2$  is determined to be  $-2\alpha/(4\pi)^3$  from (3.9), which is consistent with the RG equation (5.2).

Substituting the values of  $X_{1,2}$  and  $Y_{1,2}$  into (5.10) and (5.12), we finally obtain

$$\chi_2 = \frac{3}{4}, \quad \chi_3 = \frac{1}{3}. \quad (6.4)$$

Now, we give some comments on the universality of the function  $\chi$ . First, the value  $\chi_1 = 1/2$  is probably independent of the theory. It has been confirmed for conformally coupled massless scalar theory [6] and Yang-Mills theory [8]. Especially for Yang-Mills theory, the other coefficients of  $\chi$  may also be the same as (6.4) because the residues of pole terms satisfy almost the same RG equations as those of QED.<sup>3</sup>

Furthermore,  $\chi_1$  and  $\chi_2$  agree with those conjectured in the model of quantum gravity [10], but  $\chi_3$  unfortunately disagrees. It seems that the condition imposed to determine the action  $G_D$  in [10] may be somewhat strong. However, the difference is of higher orders and does not affect the loop calculations done there. Thus, the result is also consistent with quantum corrections including gravity.

## 7 Gravitational Effective Action

Finally, we discuss the properties of the conformal anomaly  $E_D$  and its physical implications to the effective action.

Consider the conformal variation  $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$  of the gravitational effective action  $\Gamma$  as

$$\delta_\omega \Gamma = \int d^D x \sqrt{g} \omega \left\{ \eta_1 R_{\mu\nu\lambda\sigma}^2 + \eta_2 R_{\mu\nu}^2 + \eta_3 R^2 + \eta_4 \nabla^2 R \right\}.$$

The right-hand side describes possible expressions of conformal anomalies. The Wess-Zumino consistency condition [11, 12] in  $D$  dimensions,  $[\delta_{\omega_1}, \delta_{\omega_2}] \Gamma = 0$ , gives the condition for the parameters as [10]

$$4\eta_1 + D\eta_2 + 4(D-1)\eta_3 + (D-4)\eta_4 = 0.$$

Three independent combinations satisfying this equation are given by the square of the Weyl tensor in  $D$  dimensions  $F_D$ , the usual Euler density  $G_4$

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<sup>3</sup>On the other hand, as for scalar theory, we are afraid that some uncertainty in the coupling with gravity may be left yet.

and

$$M_D = (D - 4)H^2 - 4\nabla^2 H.$$

Note that  $M_D$  corresponds to what is called the trivial conformal anomaly, but it is no longer trivial in  $D$  dimensions. The function  $E_D$  can be written in a linear combination of the usual Euler density and this function as  $E_D = G_4 + \chi(D)M_D$ .

Let us consider the four-dimensional limit in the following. Using the value (5.7), we find that the function  $E_D$  reduces to the form

$$E_4 = G_4 - \frac{2}{3}\nabla^2 R. \quad (7.1)$$

This is just the combination proposed by Riegert [13]. When the metric field is decomposed into the conformal factor and others as  $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$ , the function (7.1) satisfies the relation  $\sqrt{g}E_4 = \sqrt{\bar{g}}(4\bar{\Delta}_4\phi + \bar{E}_4)$ , where  $\sqrt{g}\Delta_4$  is a conformally invariant fourth-order differential operator for a scalar quantity defined by

$$\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\nabla^2 + \frac{1}{3}\nabla^\mu R\nabla_\mu.$$

The non-local action obtained by integrating the conformal anomaly  $b_1\sqrt{g}E_4$  over the conformal mode  $\phi$  is expressed as [13]

$$\frac{b_1}{8} \int d^4x \sqrt{g} E_4 \frac{1}{\Delta_4} E_4. \quad (7.2)$$

This action is the four-dimensional version of the Polyakov's non-local action  $\int d^2x \sqrt{g} R \Delta_2^{-1} R$  [14], where  $\Delta_2 = -\nabla^2$  is a conformally invariant operator in two dimensions.

The local part of (7.2) is given by  $b_1\sqrt{\bar{g}}(2\phi\bar{\Delta}_4\phi + \bar{E}_4\phi)$ , called the Riegert action. Thus, the kinetic term of the conformal mode is induced quantum mechanically. As similar to two dimensional gravity, the Riegert action can be quantized [15, 16, 17, 18, 19, 20] and it has been known that the combined system of the Riegert and the Weyl actions generates the BRST operator of quantum diffeomorphism that imposes for physical field operators to be, in CFT terminology, Hermitian primary scalars only [19, 20].

## 8 Conclusions

One of the significant observations that should be emphasized here is as follows. Classically, there is some uncertainty in how to choose the combinations of the fourth-order gravitational actions and their dimensionless coupling constants. When going to quantum field theory, however, it is possible to settle the problem of uncertainty by imposing the finiteness condition of renormalization.

In this paper, reconsidering the Hathrell's RG equations, we determined the expressions of the gravitational counterterms (4.1) and the conformal anomalies (4.7) for the dimensionally regularized QED in curved spacetime. We showed that at all orders of the perturbation, the independent expressions of them are only two: the square of the Weyl tensor in  $D$  dimensions  $F_D$  (2.1) and the modified Euler density  $E_D$  (4.8) whose bulk part is given by  $G_D$  (4.2). The  $D$ -dependent constant  $\chi(D)$  can be determined order by order in series of  $D - 4$ , whose first several terms were calculated explicitly as

$$\chi(D) = \frac{1}{2} + \frac{3}{4}(D - 4) + \frac{1}{3}(D - 4)^2 + o((D - 4)^3).$$

The number of the gravitational coupling constants was reduced to two. The situation will be maintained in conformally coupled theories. This is one of the results we have hope, because we think that the number of the couplings is too many to describe the dynamics of gravity.

Unlike the Hathrell's result (2.4), the final expression of conformal anomaly (4.7) has a suitable structure that when taking the  $D \rightarrow 4$  limit, the dependence on the unspecified parameters  $\mu$ ,  $a$  and  $b$  all disappears and the ambiguous  $\nabla^2 R$  term is fixed in the form  $E_4$  (7.1). Since  $\chi(4) = 1/2$  is a constant independent of the theory, the combination  $E_4$  is probably a universal expression of conformal anomaly at four dimensions.

Finally, we summarize the residues for the counterterm  $G_D$ , which were determined using the QED beta function up to 3 loop order as

$$b_1 = \frac{73}{360} \frac{1}{(4\pi)^2} - \frac{1}{6} \frac{\alpha^2}{(4\pi)^4} + \frac{25}{108} \frac{\alpha^3}{(4\pi)^5} + o(\alpha^4),$$

$$b_2 = \frac{2}{9} \frac{\alpha^3}{(4\pi)^5} + \frac{22}{135} \frac{\alpha^4}{(4\pi)^6} + o(\alpha^5).$$

Here,  $b_2$  is obtained from the expressions (5.9) and (5.11) and  $b_1$  is calculated from  $b_2$  using the RG equation (2.3) for  $b_n$ . The constant independent of  $\alpha$  in  $b_1$  cannot be determined from the RG equation, which is calculated from the direct 1-loop calculation. The  $o(\alpha^3)$  term of  $b_1$  and the  $o(\alpha^4)$  term of  $b_2$  are new results. The other residues are summarized in Appendix D. For the completion, we also add the value of the residue  $a_1$  [4],

$$a_1 = -\frac{3}{20} \frac{1}{(4\pi)^2} - \frac{7}{72} \frac{\alpha}{(4\pi)^3} + o(\alpha^2),$$

which can be calculated using (C.2) in Appendix C.

## A Derivation of The Normal Product $[F_{\mu\nu} F^{\mu\nu}]$

We here briefly summarize how to derive the expression of the normal product  $[F_{\mu\nu} F^{\mu\nu}]$  [7].

First, we consider the finite quantity obtained by applying  $\xi \partial / \partial \xi$  to the renormalized correlation function  $\langle \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_k) \rangle$ . We then obtain

$$\left\langle \int d^D x \sqrt{g} \left\{ \frac{1}{\xi} (\nabla^\mu A_\mu)^2 - [E_\psi] \xi \frac{\partial}{\partial \xi} \log Z_2 \right\} \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_k) \right\rangle = \text{finite.} \quad (\text{A.1})$$

Next, in order to obtain a finite expression including  $F_{0\mu\nu} F_0^{\mu\nu}$ , we consider the finite quantity derived by applying  $\alpha \partial / \partial \alpha$  to the renormalized correlation function. The  $\alpha$ -dependences of various bare parameters are calculated as  $\alpha \partial e_0 / \partial \alpha = (D-4)e_0/2\beta$ ,  $\alpha \partial \xi_0 / \partial \alpha = \xi_0 \bar{\beta} / \beta$ ,  $\alpha \partial (\log Z_3) / \partial \alpha = \bar{\beta} / \beta$  and  $\alpha \partial (\log Z_2) / \partial \alpha = [\gamma_2 + \bar{\beta} \xi \partial (\log Z_2) / \partial \xi] / \beta$  for the QED sector. For the gravity sector, we obtain  $\alpha \partial a_0 / \partial \alpha = -\mu^{D-4} [(D-4)L_a + \bar{\beta}_a] / \beta$  and similar equations

for  $b_0$  and  $c_0$ . Using these, we finally obtain

$$\begin{aligned} & \left\langle \int d^D x \sqrt{g} \left\{ \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} - \frac{\bar{\gamma}_2}{2\beta} E_{0\psi} + \frac{D-4}{\beta} \mu^{D-4} \left[ \left( L_a + \frac{\bar{\beta}_a}{D-4} \right) F_D \right. \right. \right. \\ & \left. \left. \left. + \left( L_b + \frac{\bar{\beta}_b}{D-4} \right) G_4 + \left( L_c + \frac{\bar{\beta}_c}{D-4} \right) H^2 \right] \right\} \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_k) \right\rangle \\ & = \text{finite.} \end{aligned}$$

Here, we use the fact that  $N_A$  and  $N_\psi$  can be replaced with the volume integrals of the equation-of-motion operators  $E_{0A}$  and  $E_{0\psi}$  (B.1) in the correlation function. The interaction term  $e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu}$  is also eliminated by using  $E_{0A}$  and then the kinetic term of gauge field appears. The finite combination (A.1) and the apparently finite quantity put away to the right-hand side.

This equation means that the inside of the bracket  $\{ \}$  is the normal ordered quantity up to total divergences. Here, noting that  $(D-4)/\beta = 1 + \sum \text{poles}$  (2.2), it has the structure of the normal product mentioned in the text and thus it is identified with  $[F_{\mu\nu} F^{\mu\nu}]/4$ . Since the candidate for the total divergence term is only  $\nabla^2 H$ , we obtain

$$\begin{aligned} \frac{1}{4} [F_{\mu\nu} F^{\mu\nu}] &= \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} - \frac{\bar{\gamma}_2}{2\beta} E_{0\psi} + \frac{D-4}{\beta} \mu^{D-4} \left[ \left( L_a + \frac{\bar{\beta}_a}{D-4} \right) F_D \right. \\ & \quad \left. + \left( L_b + \frac{\bar{\beta}_b}{D-4} \right) G_4 + \left( L_c + \frac{\bar{\beta}_c}{D-4} \right) H^2 - \frac{4(\sigma + L_\sigma)}{D-4} \nabla^2 H \right]. \end{aligned} \tag{A.2}$$

Here,  $\sigma$  is a finite function of  $\alpha$  and  $L_\sigma$  is the pure-pole term, which are defined through this equation. These quantities are determined by imposing other finiteness conditions. The results are given in the text.

## B The Equation-of-Motion Operators

The equation-of-motion operators for gauge and fermion fields are defined,



respectively, by

$$\begin{aligned} E_{0A} &= \frac{1}{\sqrt{g}} A_{0\mu} \frac{\delta S}{\delta A_{0\mu}} = A_{0\mu} \nabla_\nu F_0^{\mu\nu} - e_0 \bar{\psi}_0 \gamma^\mu A_{0\mu} \psi_0 - \frac{1}{\xi_0} A_{0\mu} \nabla^\mu \nabla^\nu A_{0\nu}, \\ E_{0\psi} &= \frac{\delta S}{\delta \chi} \equiv \frac{1}{\sqrt{g}} \left( \bar{\psi}_0 \frac{\delta S}{\delta \bar{\psi}_0} + \psi_0 \frac{\delta S}{\delta \psi_0} \right) = 2i \bar{\psi}_0 \overleftrightarrow{D} \psi_0, \end{aligned} \quad (\text{B.1})$$

where covariant derivative with arrow  $\overleftrightarrow{D}_\mu$  is defined by replacing  $\partial_\mu$  in  $D_\mu$  with  $(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)/2$ .

Although the equation-of-motion operators are written in terms of the bare fields, they are finite in correlation functions. It is demonstrated in the path integral formalism as follows. Carrying out an integration-by-part, we obtain the following relations:

$$\begin{aligned} \left\langle E_{0A}(x) \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \right\rangle &= \sum_{j=1}^{N_A} \frac{1}{\sqrt{g}} \delta^D(x - x_j) \left\langle \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \right\rangle, \\ \left\langle E_{0\psi}(x) \prod_{j=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_j) \right\rangle &= \sum_{j=1}^{N_\psi} \frac{1}{\sqrt{g}} \delta^D(x - x_j) \left\langle \prod_{j=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_j) \right\rangle. \end{aligned} \quad (\text{B.2})$$

Here, note that there is no term from functional differentials at the same point, because it is dimensionally regularized to zero such as  $\delta A_\mu(x)/\delta A_\nu(x) = \delta^\mu_\nu \delta^D(0) = 0$ . The right-hand sides are obviously finite and thus the left-hand sides are also finite. So, the equation-of-motion operators can be written in terms of the normal products as

$$E_{0A} = [E_A], \quad E_{0\psi} = [E_\psi].$$

From (B.2),  $\int d^D x \sqrt{g} E_{0A}$  and  $\int d^D x \sqrt{g} E_{0\psi}$  can be replaced with the numbers  $N_A$  and  $N_\psi$ , respectively, in correlation functions.

## C Finiteness Conditions for Two and Three-Point Functions

The energy-momentum tensor is defined by  $\theta^{\mu\nu} = (2/\sqrt{g})\delta S/\delta g_{\mu\nu}$  and its

trace is denoted by  $\theta = \theta^\mu{}_\mu = \delta S / \delta \Omega$ . In flat space, the energy-momentum tensor is given by

$$\theta^{\mu\nu}|_{\text{flat}} = -F_0^{\mu\lambda} F_{0\lambda}^\nu + \frac{1}{4} g^{\mu\nu} F_{0\lambda\sigma} F_0^{\lambda\sigma} - \frac{i}{2} \bar{\psi}_0 \left( \gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu - 2g^{\mu\nu} \overleftrightarrow{D} \right) \psi_0$$

and its trace is

$$\theta|_{\text{flat}} = (D-4) \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + (D-1) i \bar{\psi}_0 \overleftrightarrow{D} \psi_0.$$

Here, we disregard the term of gauge-fixing origin because it gives no contribution in physical correlation functions.

Since the partition function is finite, its gravitational variations are also finite. Thus, carrying out the variation two times, we obtain

$$\langle \theta^{\mu\nu}(x) \theta^{\lambda\sigma}(y) \rangle - \frac{2}{\sqrt{g(y)}} \left\langle \frac{\delta \theta^{\mu\nu}(x)}{\delta g_{\lambda\sigma}(y)} \right\rangle = \text{finite}.$$

Taking the flat space limit and going to momentum space, we obtain the following condition:

$$\langle \theta^{\mu\nu}(p) \theta^{\lambda\sigma}(-p) \rangle_{\text{flat}} - a_0 A^{\mu\nu, \lambda\sigma}(p) - c_0 C^{\mu\nu, \lambda\sigma}(p) = \text{finite},$$

where the functions  $A^{\mu\nu, \lambda\sigma}$  and  $C^{\mu\nu, \lambda\sigma}$  are defined by

$$\begin{aligned} A^{\mu\nu, \lambda\sigma}(p) &= \frac{4(D-3)}{D-2} \left[ p^4 (\delta^{\mu\lambda} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\lambda}) - p^2 (\delta^{\mu\lambda} p^\nu p^\sigma + \delta^{\mu\sigma} p^\nu p^\lambda + \delta^{\nu\lambda} p^\mu p^\sigma \right. \\ &\quad \left. + \delta^{\nu\sigma} p^\mu p^\lambda) + 2p^\mu p^\nu p^\lambda p^\sigma \right] - \frac{8(D-3)}{(D-1)(D-2)} \left[ p^4 \delta^{\mu\nu} \delta^{\lambda\sigma} \right. \\ &\quad \left. - p^2 (\delta^{\mu\nu} p^\lambda p^\sigma + \delta^{\lambda\sigma} p^\mu p^\nu) + p^\mu p^\nu p^\lambda p^\sigma \right], \\ C^{\mu\nu, \lambda\sigma}(p) &= \frac{8}{(D-1)^2} \left[ p^4 \delta^{\mu\nu} \delta^{\lambda\sigma} - p^2 (\delta^{\mu\nu} p^\lambda p^\sigma + \delta^{\lambda\sigma} p^\mu p^\nu) + p^\mu p^\nu p^\lambda p^\sigma \right], \end{aligned}$$

which are derived from the  $F_D$  and  $H^2$  terms in the action, respectively. The contracting the indices of the energy-momentum tensor, we obtain

$$\langle \theta(p) \theta(-p) \rangle_{\text{flat}} - 8c_0 p^4 = \text{finite} \quad (\text{C.1})$$

and

$$\langle \theta^{\mu\nu}(p) \theta_{\mu\nu}(-p) \rangle_{\text{flat}} - 4(D-3)(D+1)a_0 p^4 - \frac{8}{D-1}c_0 p^4 = \text{finite}. \quad (\text{C.2})$$

And also from the variation of the partition function with respect to  $\Omega$  three times, we obtain

$$\begin{aligned} & \langle \theta(x) \theta(y) \theta(z) \rangle - \left\langle \frac{\delta \theta(x)}{\delta \Omega(y)} \theta(z) \right\rangle - \left\langle \frac{\delta \theta(y)}{\delta \Omega(z)} \theta(x) \right\rangle - \left\langle \frac{\delta \theta(z)}{\delta \Omega(x)} \theta(y) \right\rangle \\ & + \left\langle \frac{\delta \theta(x)}{\delta \Omega(y) \delta \Omega(z)} \right\rangle = \text{finite}. \end{aligned} \quad (\text{C.3})$$

## D Values of The Residues

Substituting the values (6.1) and (6.2) into the expression of  $x_n$  (5.1), we obtain

$$\begin{aligned} x_1(\alpha) &= -\frac{1}{2} \frac{1}{(4\pi)^2} + \frac{5}{3} \frac{\alpha}{(4\pi)^3} + o(\alpha^2), \\ x_2(\alpha) &= -\frac{4}{3} \frac{\alpha}{(4\pi)^3} + \frac{16}{9} \frac{\alpha^2}{(4\pi)^4} + o(\alpha^3), \\ x_3(\alpha) &= \frac{8}{9} \frac{\alpha^3}{(4\pi)^5} - \frac{40}{27} \frac{\alpha^4}{(4\pi)^6} + o(\alpha^5), \\ x_4(\alpha) &= -\frac{64}{45} \frac{\alpha^4}{(4\pi)^6} - \frac{224}{243} \frac{\alpha^5}{(4\pi)^7} + o(\alpha^6). \end{aligned}$$

Substituting (6.1), (6.2) and (6.3) into the expression of  $y_n$  (5.3), we obtain

$$\begin{aligned} y_1(\alpha) &= -\frac{1}{2} \frac{1}{(4\pi)^2} + \frac{11}{6} \frac{\alpha}{(4\pi)^3} + o(\alpha^2), \\ y_2(\alpha) &= -2 \frac{\alpha}{(4\pi)^3} + \frac{184}{27} \frac{\alpha^2}{(4\pi)^4} + o(\alpha^3), \\ y_3(\alpha) &= -\frac{16}{9} \frac{\alpha^2}{(4\pi)^4} + \frac{740}{81} \frac{\alpha^3}{(4\pi)^5} + o(\alpha^4), \\ y_4(\alpha) &= -\frac{32}{45} \frac{\alpha^4}{(4\pi)^6} - \frac{9712}{1215} \frac{\alpha^5}{(4\pi)^7} + o(\alpha^6), \\ y_5(\alpha) &= \frac{512}{405} \frac{\alpha^5}{(4\pi)^7} + \frac{416128}{25515} \frac{\alpha^6}{(4\pi)^8} + o(\alpha^7). \end{aligned}$$

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